Practical 3 4600
State Space Output Closed-Loop Feedback Controll

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Contents

1 Task

The plant consists of a permanent magnet dc electric servo motor/gearbox with angular position feedback. Open loop test have shown that the relationship between applied voltage (volts) and resultant rotational speed (rad/sec) can be described by the transfer function.

$$G(s) = \frac{1}{s+16} \tag{1}$$

The controller which should be designed has to meet the following performance criteria:

- Steady state error to a step change in desired position = zero
- Maximum percentage overshoot to a step change in desired position = 4.4
- Maximum settling time allowed following a step change in desired position = 0.25sec

The output of the Plants Transfer function is in (rad/sec) but the motor gives only an angular posiition as feedback therefore the Transferfunction (equation 1) must be Integrated. That means in Laplace domain it must be multiplicated with $\frac{1}{s}$.

$$G(s) = \frac{1}{s(s+16)} = \frac{1}{s^2+16s} \tag{2}$$

2 Solution

2.1 State space model

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+16s} \tag{3}$$

$$Y(s)s^2 + Y(s)16s = U(s) \tag{4}$$

in the time domain

$$\ddot{y} + 16\dot{y}(t) = u(t) \tag{5}$$

States will are defined as follwing:

$$y_1(t) = y(t) \rightarrow \dot{y_1}(t) = \dot{y}(t) = y_2 \tag{6}$$
$$y_2(t) = \dot{y}(t) \rightarrow \dot{y_2} = \ddot{y}(t) \tag{7}$$

with the states (eq 7) equation 5 can be written as

$$\dot{y_2} + 16y_2(t) = u(t) \tag{8}$$
$$\dot{y_2} = -16y_2(t) + u(t) \tag{9}$$

In matrix form

$$\begin{bmatrix} \dot{y_1}(t) \\ \dot{y_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -16 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{10}$$
$$y(t) = [1\ 0] \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{11}$$

This gives the following results for the system matrixes

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -16 \end{bmatrix} \tag{12}$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{13}$$
$$C = [1\ 0] \tag{14}$$

2.2 Controllability

The controllability matrix CM must be derived to check the system for controllability

$$CM = [A\ AB] = \begin{bmatrix} 0 & 1 \\ 1 & -16 \end{bmatrix} \rightarrow det \neq 0$$

Because of CM is a requllar matrix the system is controllable. This means it is possible to build a controller for this system.

2.3 Controller design

The percentage overshoot for a step change input is given as

$$P.O. = 100e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}} \tag{15}$$
$$\zeta = \frac{-\ln(\frac{P.O.}{100})}{\sqrt{\pi^2 + \ln(\frac{P.O.}{100})^2}} \tag{16}$$

And the settling time can be derived by

$$T_s = \frac{4}{\zeta\omega_n} \tag{17}$$
$$\omega_n = \frac{4}{T_s\zeta} \tag{18}$$

With equation 16 and 18 both values of ω_n and ζ can be derived as following.

$$\zeta = 0.705 \qquad \omega_n = 22.7$$

Using the standard form of a second order system (eq. 19) the desired pole positions for the above derive ζ and ω_n can be calcuated as following (eq, 20).

$$
\begin{aligned}
G &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} && (19)\\
0 &= s^2 + 2\zeta\omega_n s + \omega_n^2 \\
\rightarrow \qquad s_{1/2} &= -\zeta\omega_n \pm sqrt(\zeta\omega_n)^2 - \omega_n^2 && (20)
\end{aligned}
$$

The desired Pole locations for the controller are $\underline{-16 \pm 16.1j}$.

The system is already in canonical form therefor is it possible to use the pole placement techniques to design a controller with the desired pole locations .

$$CLCE = (s + 16 - 16.1j)(s + 16 + 16.1j) = s^2 + 32s + 512 \tag{21}$$

The Feedback Controller will have the following transfer function.

$$
\begin{aligned}
\underline{w}(t) &= -F\underline{x}(t) + r(t) && (22)\\
\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} &= -\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + r(t) && (23)
\end{aligned}
$$

The closed Loop matrix

$$A_{CL} = A - BF = \begin{bmatrix} 0 & 1 \\ -F_1 & -16 - F_2 \end{bmatrix}$$

can be derived with

$$
\begin{aligned}
det\,(sI - (A - BF)) &= s^2 - (-16 - F_2)s - (-F_1) \\
&= s^2 + (16 + F_2)s + F_1) && (24)\\
& && (25)
\end{aligned}
$$

The equation 24 must be equal equation 21

$$s^2 + (16 + F_2)s + F_1) = s^2 + 32s + 512$$

$16 + F_2 = 32 \qquad \rightarrow \qquad F_2 = 16 \;\; F_1 = 512$

Therefore the resulting transfer function for the controller is

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = -\begin{bmatrix} 512 & 16 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + r(t)$$

and the desired closed loop matrix

$$A_{CL} = A - BF = \begin{bmatrix} 0 & 1 \\ -512 & -32 \end{bmatrix}$$

In figure 1 the step response of the model is schown. As it can be seen the performance criteria met.

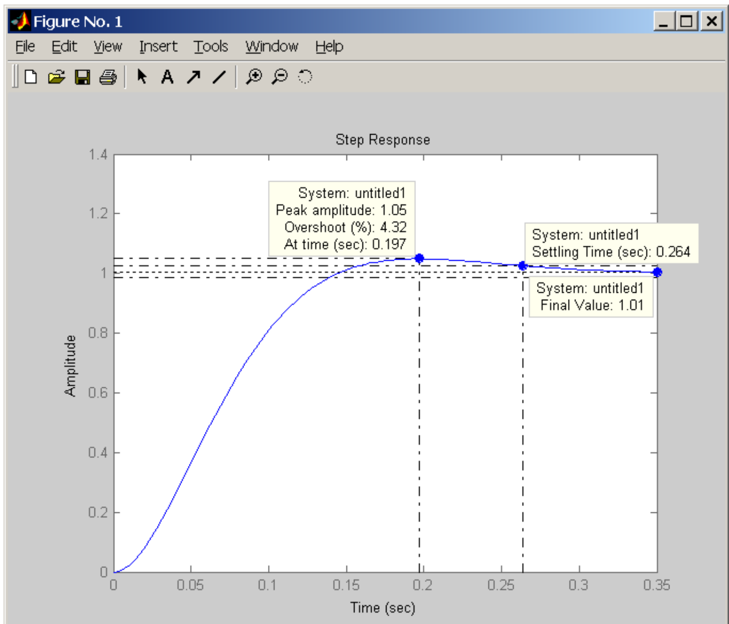

Figure 1: Step response

2.4 Observability

If the system is observable the observability matrix is regular

$$OM = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rightarrow \qquad det(OM) \neq 0$$

Because the observability matrix is regular the system is observable. This means it exists a path between each state and the output. Therefore it it is possible to estimate a state variable of the system.

2.5 Observer construction

The transfer function of the observer will be

$$\underline{\dot{x}^*(t)} = A^*\underline{x^*}(t) + B^*\underline{u^*}(t)$$

$$\underline{\dot{y}^*(t)} = C^*\underline{x^*}(t)$$

where

$$\underline{x^*} = \begin{bmatrix} \underline{x}(t) \\ \underline{\hat{x}}(t) \end{bmatrix}, \quad A^* = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix}, \quad B^* = \begin{bmatrix} B \\ B \end{bmatrix}, \quad C^* = \begin{bmatrix} C & 0 \end{bmatrix}$$

with the initial condition

$$\underline{x^*}(0) = \begin{bmatrix} \underline{x}(0) \\ \underline{\hat{x}}(0) \end{bmatrix}$$

The Observer should have a faste dynamic as the system itself. Because we have choosen a maximum settling time for the system of 0.25sec we choose 0.1sec for the observer. And the same maximum overshoot as the system.

$$\omega_n = 56.7 \qquad \zeta = 0.7$$

Therefore the desired pole locations for the observer are $-40 \pm 40j$.

$$CE_{obs} = S^2 + 80s + 3128.5$$

and a $L = \begin{bmatrix} 64 \\ 2176 \end{bmatrix}$ is derived.

$$\underline{\dot{x}^*(t)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -16 & 0 & 0 \\ 64 & 0 & -64 & 1 \\ 2176 & 0 & -2176 & -16 \end{bmatrix} \underline{x^*}(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \underline{u^*}(t)$$

$$\underline{\dot{y}^*(t)} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \underline{x^*}(t)$$

The modeled system (simulink) is shown in figure 2. In figure 3 is the response of the observer shown.

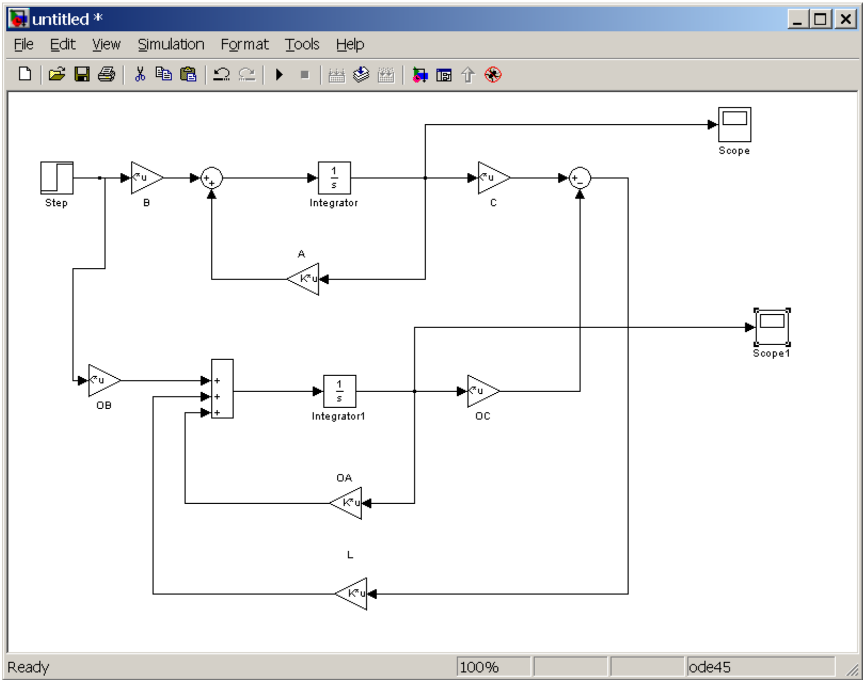

Figure 2: Simulink model

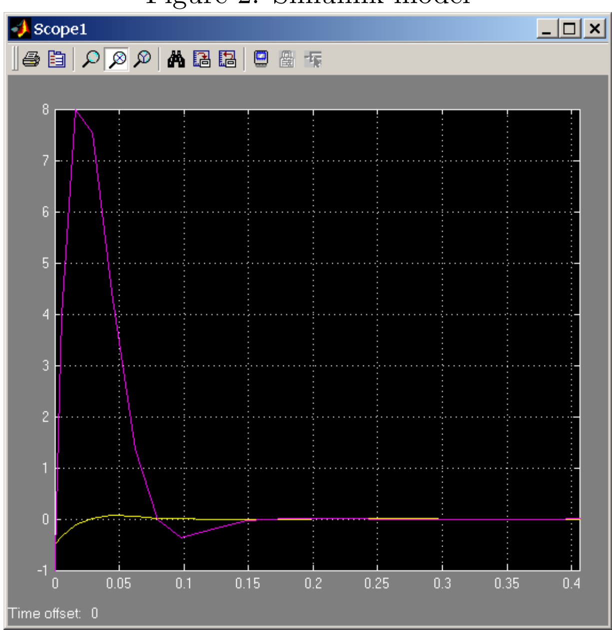

Figure 3: Step response - observer